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RESONANCE OSCILLATIONS OF A SPECIAL DOUBLE PENDULUM

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Resonance oscillations of a mechanical system are investigated, and peculiarities in its behavior are explained. The oscillations of conservative systems with two degrees of freedom under internal resonance are examined in [1−5]. A certain addition to the existing asymptotic methods in the theory of nonlinear oscillations is proposed in the last paper by Struble; the results of this paper are utilized below.

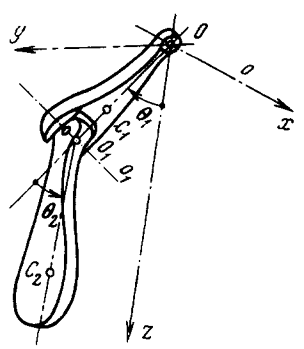

Fig. 1

1. Let us consider a system of two successively connected physical pendulums (Fig. 1). The first rotates around a horizontal axis o, and the second around an axis o_1 belonging to the first pendulum and perpendicular to o. In the equilibrium position o_1 is horizontal. Let C_1 and C_2 denote the centers of gravity of the two physical pendulums; M_1 and M_2 their masses; O_1 the intersection of the line OC_1 with the o_1-axis; I_1 the moment of inertia of the first pendulum relative to an axis passing through C_1 and parallel to o; I_{22} the moment of inertia of the second pendulum relative to the axis passing through C_2 and parallel to o_1; I_{23} passing through C_2 and O_1; I_{21} passing through C_2 and perpendicular to the other two axes. We shall consider I_{21}, I_{22} and I_{23} to be the principal central moments of inertia of the second pendulum; let θ_1 be the deflection of the first pendulum from the oz-axis, and θ_2 the deflection of the second pendulum from the OC_1-axis; let us set

$$OC_1 = a_1,\ O_1C_2 = a_2,\ OO_1 = b_1$$

In this notation we have:

for the system kinetic energy

$$T = {}^1/_2 [I_1 + M_1 a_1^2 + I_{21} \cos^2\theta_2 + I_{23} \sin^2\theta_2 + M_2 (b_1 + a_2 \cos\theta_2)^2]\, \theta_1^{\cdot 2} + \\ + {}^1/_2 (I_{22} + M_2 a_2^2) \theta_2^{\cdot 2} \tag{1.1}$$

for the system potential energy

$$\Pi = - g (M_1 a_1 + M_2 b_1) \cos\theta_1 - g M_2 a_2 \cos\theta_1 \cos\theta_2 \tag{1.2}$$

Taking account of (1.1) and (1.2), we find the differential equation of motion

$$[I_1 + M_1 a_1^2 + I_{21} \cos^2\theta_2 + I_{23} \sin^2\theta_2 + M_2 (b_1 + a_2 \cos\theta_2)^2]\ \theta_1^{\cdot\cdot} - \\ - [(I_{21} - I_{23} + M_2 a_2^2) \sin 2\theta_2 + 2 M_2 b_1 a_2 \sin\theta_2] \theta_1^{\cdot} \theta_2^{\cdot} + \\ + g [M_1 a_1 + M_2 (b_1 + a_2 \cos\theta_2)] \sin\theta_1 = 0 \\ (I_{22} + M_2 a_2^2)\, \theta_2^{\cdot\cdot} + [M_2 a_2 b_1 \sin\theta_2 + {}^1/_2 (I_{21} - I_{23} + M_2 a_2^2)\ \sin\ 2\theta_2]\ \theta_1^{\cdot 2} + \\ + g M_2 a_2 \cos\theta_1 \sin\theta_2 = 0 \tag{1.3}$$

Let us set

$$\theta_1 = \varepsilon z_1, \qquad \theta_2 = \varepsilon z_2, \qquad \varepsilon \text{ — small parameter} \tag{1.4}$$

After some manipulation, we obtain from (1.3)

$$[I_1 + M_1 a_1^2 + I_{21} + M_2 (b_1 + a_2)^2]\, z_1^{\cdot\cdot} + g [M_1 a_1 + M_2 (b_1 + a_2)]\, z_1 = \varepsilon^2 F_1 \\ (I_{22} + M_2 a_2^2)\ z_2^{\cdot\cdot} + g\ M_2 a_2 z_2 = \varepsilon^2 F_2 \\ F_1 = [I_{21} - I_{23} + M_2 a_2 (b_1 + a_2)]\, (z_1^{\cdot\cdot} z_2^2 + 2 z_1^{\cdot} z_2 z_2^{\cdot}) + {}^1/_6 g [M_1 a_1 + \\ + M_2 (b_1 + a_2)]\, z_1^3 + {}^1/_2 g\, M_2 a_2 z_1 z_2^2 + \varepsilon^2 (\ldots) + \ldots \\ F_2 = - [I_{21} - I_{23} + M_2 a_2 (b_1 + a_2)]\, z_1^{\cdot 2} z_2 + {}^1/_6 g M_2 a_2 (3 z_1^2 z_2 + z_2^3) + \varepsilon^2\ (\ldots) + \ldots \tag{1.5}$$

Let us introduce dimensionless time

$$\tau = t \frac{g M_2 a_2}{I_{22} + M_2 a_2^2} \tag{1.6}$$

We then find from (1.5)

$$z_1'' + \beta^2 z_1 = \varepsilon^2 [ab (z_1'' z_2^2 + 2 z_1' z_2 z_2') + {}^1/_6 \beta^2 z_1^3 + {}^1/_2 a z_1 z_2^2] \\ z_2'' + z_2 = \varepsilon^2 [- b z_1'^2 z_2 + {}^1/_6 z_2^3 + {}^1/_2 z_1^2 z_2] \\ a = \frac{I_{22} + M_2 a_2^2}{I_1 + I_{21} + M_1 a_1^2 + M_2 (b_1 + a_2)^2}, \qquad b = \frac{I_{21} - I_{23} + M_2 a_2 (b_1 + a_2)}{I_{22} + M_2 a_2^2} \\ \beta^2 = \frac{(I_{22} + M_2 a_2^2) [M_1 a_1 + M_2 (b_1 + a_2)]}{[I_1 + I_{21} + M_1 a_1^2 + M_2 (b_1 + a_2)^2]\ M_2 a_2 \theta} \tag{1.7}$$

where derivatives with respect to τ and terms containing ε in powers higher than the second are discarded.

2. Now, let us examine the system (1.7). Let us seek the solution in the form

$$z_1 = A \cos(\beta\tau - \psi) + \varepsilon^2 z_{12} + \varepsilon^4 z_{14} + \ldots \\ z_2 = B \cos\ (\tau - \psi)\ + \varepsilon^2 z_{22} + \varepsilon^4 z_{24} + \ldots \tag{2.1}$$

where A, B, φ, ψ are slowly varying functions of τ. Substituting (2.1) into (1.7) we find

$$(A'' + 2\beta A \varphi' - A \varphi'^2)\ \cos\ (\beta\tau - \varphi) + \\ + (A \varphi'' - 2\beta A' + 2 A' \varphi')\ \sin\ (\beta\tau - \varphi) + \varepsilon^2 (z_{12}'' + \beta^2 z_{12}) = \\ = \varepsilon^2 \{[{}^1/_4 A B^2 a (1 - 2 b\beta^2) + {}^1/_8 \beta^2 A^3] \cos\ (\beta\tau - \varphi) + {}^1/_{24} \beta^2 A^3 \cos\ (3\beta\tau - 3\varphi) + \\ + {}^1/_8 a A B^2 (1 + 4 b\beta - 2 b\beta^2) \cos [(\beta - 2)\, \tau - \varphi + 2\psi] + \\ + {}^1/_8 a A B^2 (1 - 4 b\beta - 2 b\beta^2) \cos [(\beta + 2)\, \tau - \varphi - 2\psi]\} \\ (B'' + 2 B \psi' - B \psi'^2) \cos (\tau - \psi) + (B \psi'' - 2 B' + 2 B' \psi') \sin\ (\tau - \psi) + \\ + \varepsilon^2 (z_{22}'' + z_{22}) = \varepsilon^2 \{[{}^1/_4 A^2 B\ (1 - 2 b\beta^2) + {}^1/_8 B^3]\ \cos\ (\tau - \psi) + \\ + {}^1/_{24} B^3 \cos\ (3\tau - 3\psi) + {}^1/_8 A^2 B\ (1 + 2 b\beta^2)\ \cos\ [(2\beta - 1)\, \tau - 2\varphi + \psi] + \\ + {}^1/_8 A^2 B\ (1 + 2\ b\beta^2)\ \cos\ [(2\beta + 1)\, \tau - 2\varphi - \psi]\} \tag{2.2}$$

Here terms containing ε to degrees higher than the second have been discarded. From (2.2) we deduce

$$A'' + 2\beta A\varphi' - A\varphi'^2 = \varepsilon^2 [{}^1/_4\, aAB^2(1-2b\beta^2) + {}^1/_8\beta^2A^3], \quad A\varphi'' - 2\beta A' + 2A'\varphi' = 0$$
$$B'' + 2B\psi' - B\psi'^2 = \varepsilon^2 [{}^1/_4\, A^2B\, (1-2b\beta^2) + {}^1/_8\, B^3], \quad B\psi'' - 2B' + 2B'\psi' = 0$$
$$z_{12}'' + \beta^2 z_{12} = {}^1/_8\, AB^2 a\, (1 + 4b\beta - 2b\beta^2) \cos\, [(\beta - 2)\, \tau - \varphi + 2\psi] + \qquad (2.3)$$
$$+ {}^1/_{24}\, \beta^2 A^3 \cos\, (3\beta\tau - 3\varphi) + {}^1/_8\, AB^2 a\, (1 - 4b\beta - 2b\beta^2) \cos\, [(\beta + 2)\, \tau - \varphi - 2\psi]$$
$$z_{22}'' + z_{22} = {}^1/_8\, A^2B\, (1 + 2b\beta^2) \cos\, [(2\beta - 1)\, \tau - 2\varphi + \psi] + {}^1/_{24}\, B^3 \cos\, (3\tau - 3\psi) +$$
$$+ {}^1/_8\, A^2B\, (1 + 2\, b\beta^2) \cos\, [(2\beta + 1)\, \tau - 2\varphi - \psi] \qquad (2.4)$$

Equations (2.3) are called variational, and (2.4) the perturbation equations [2–5]. From (2.3) we easily obtain

$$d\varphi\, /\, d\tau = {}^1/_{16}\, \beta^{-1}\varepsilon^2\, [2a\, (1-2\, b\beta^2)\, B^2 + \beta^2 A^2], \qquad dA\, /\, d\tau = 0$$
$$d\psi\, /\, d\tau = {}^1/_{16}\, \varepsilon^2\, [2\, (1-2b\beta^2)\, A^2 + B^2], \qquad dB\, /\, d\tau = 0 \qquad (2.5)$$

The solution of this system of equations is

$$\varphi = {}^1/_{16}\, \beta^{-1}\, [2a(1 - 2b\beta^2)\, B_0^2 + \beta^2 A_0^2]\, \varepsilon^2\tau + \varphi_0$$
$$\psi = {}^1/_{16}\, [2\, (1-2b\beta^2)\, A_0^2 + B_0^2]\, \varepsilon^2\tau + \psi_0 \qquad (2.6)$$

Here φ_0, ψ_0, A_0, B_0 are constants of integration. From (2.4) we find for $\beta \neq 1$

$$z_{12} = {}^1/_{32}\, (\beta - 1)^{-1}\, aAB^2\, (1 + 4b\beta - 2b\beta^2) \cos\, [(\beta - 2)\, \tau - \varphi + 2\psi] - \qquad (2.7)$$
$$- {}^1/_{192}\, A^3 \cos\, (3\, \beta\tau - 3\varphi) -$$
$$- {}^1/_{32}\, (\beta + 1)^{-1}\, aAB^2\, (1-4b\beta - 2b\beta^2) \cos\, [(\beta + 2)\, \tau - \varphi - 2\psi]$$

$$z_{22} = {}^1/_{32}\, \beta^{-1}\, (1 - \beta)^{-1} A^2B(1 + 2b\beta^2) \cos\, [(2\beta - 1)\, \tau - 2\varphi + \psi] -$$

$$- {}^1/_{192}\, B^3 \cos\, (3\tau - 3\psi) - {}^1/_{32}\, \beta^{-1}(1 + \beta)^{-1} A^2B(1 + 2b\beta^2) \cos\, [(2\beta + 1)\tau - 2\, \varphi - \psi]$$

Thus, for $\beta \neq 1$ the solution of the system (1.7) to the accuracy of terms containing ε to powers not higher than the second has the form (2.1), where A, B, φ, ψ, z_{12}, z_{22} are defined by (2.6) and (2.7).

3. Now, let us also examine the resonance solution when $\beta \approx 1$ and $\beta = 1$. Utilizing the identities

$$\cos\, [(\beta - 2)\, \tau - \varphi + 2\psi] = \cos\, [2(\beta - 1)\, \tau - 2\, \varphi + 2\psi] \cos\, (\beta\tau - \varphi) + \sin\, [2(\beta - 1)\times$$
$$\times\tau - 2\psi + 2\varphi] \sin\, (\beta\tau - \varphi)$$

$$\cos\, [(2\beta - 1)\, \tau - 2\varphi + \psi] = \cos\, [2(\beta - 1)\, \tau - 2\varphi + 2\psi] \cos\, (\tau - \psi) - \sin\, [2\, (\beta - 1)\times$$
$$\times\tau - 2\varphi + 2\psi] \sin\, (\tau - \psi) \qquad (3.1)$$

we find from (2.2) in place of the (2.3) and (2.4)

$$A'' + 2\beta A\varphi' - A\varphi'^2 = \varepsilon^2\, \{{}^1/_8\, \beta^2 A^3 + {}^1/_4\, aAB^2\, (1 - 2b\beta^2) +$$
$$+ {}^1/_8\, aAB^2\, (1 + 4b\beta - 2b\beta^2) \cos\lambda\}$$
$$- 2\beta A' + A\varphi'' + 2A'\varphi' = \varepsilon^2\, {}^1/_8\, aAB^2\, (1 + 4b\beta - 2b\beta^2) \sin\lambda$$

$$B'' + 2B\psi' - B\psi'^2 = \varepsilon^2\, \{{}^1/_8\, B^3 + {}^1/_4\, A^2B\, (1-2b\beta^2) + {}^1/_8\, A^2B\, (1 + 2b\beta^2) \cos\lambda\}$$
$$- 2B' + B\psi'' + 2B'\psi' = - \varepsilon^2\, {}^1/_8\, A^2B\, (1 + 2b\beta^2) \sin\lambda \qquad (3.2)$$

$$\lambda = 2\, (\beta - 1) - 2\varphi + 2\psi \qquad (3.3)$$

$$z_{12}'' + \beta^2 z_{12} = {}^1/_8\, aAB^2\, (1 - 4b\beta - 2b\beta^2) \cos\, [(\beta + 2)\, \tau - \varphi - 2\psi] +$$
$$+ {}^1/_{24}\, \beta^2 A^3 \cos\, (3\beta\tau - 3\varphi) \qquad (3.4)$$
$$z_{22}'' + z_{22} = {}^1/_8 A^2B\, (1 + 2b\beta^2) \cos\, [(2\beta + 1)\, \tau - 2\varphi - \psi] + {}^1/_{24}\, B^3 \cos\, (3\tau - 3\psi)$$

From (3.4) we obtain

$$z_{12} = -{}^1/_{32}\,(\beta+1)^{-1}\,aAB^2\,(1-4b\beta-2b\beta^2)\cos[(\beta+2)\tau-\varphi-2\psi] - \\ -{}^1/_{192}\,A^3\cos(3\beta\tau-3\varphi) \qquad (3.5)$$

$$z_{22} = -{}^1/_{32}\,\beta^{-1}(\beta+1)^{-1}A^2B\,(1+2b\beta^2)\cos[(2\beta+1)\tau-2\varphi-\psi] - {}^1/_{192}\,B^3\cos(3\tau-3\psi)$$

It is easy to confirm that any solution of the system

$$dA/d\tau = -{}^1/_{16}\,\varepsilon^2\,aAB^2\,(1+4b\beta-2b\beta^2)\sin\lambda \qquad (3.6)$$

$$d\varphi/d\tau = -{}^1/_{16}\varepsilon^2\beta^{-1}[\beta^2A^2+2a(1-2b\beta^2)B^2] + \varepsilon^2\,{}^1/_{16}\,\beta\;a(1+4b\beta-2b\beta^2)B^2\cos\lambda$$

$$dB/d\tau = {}^1/_{16}\varepsilon^2\beta^{-1}A^2B\,(1+2b\beta^2)\sin\lambda$$

$$d\psi/d\tau = {}^1/_{16}\,\varepsilon^2\,[B^2+2(1-2b\beta^2)A^2] + {}^1/_{16}\,\varepsilon^2\beta^{-1}(1+2b\beta^2)A^2\cos\lambda$$

satisfies the system (3.2) to the accuracy of second order terms in ε.

After eliminating τ and integrating, we find from the first and third equations of the system (3.6)

$$\sigma^2A^2+B^2=\varkappa^2 \qquad \left(\sigma^2=\frac{1+2b\beta^2}{a\beta(1+4b\beta-2b\beta^2)}\right) \qquad (3.7)$$

where $\varkappa^2$ is a constant of integration.

From (3.3), (3.6) and (3.7) we obtain an autonomous system in the two variables A and λ

$$\frac{dA}{du} = -\frac{1}{2}A\left(\frac{\varkappa^2}{\sigma^2}-A^2\right)\sin\lambda, \quad u=\frac{1+2b\beta^2}{8\beta}\varepsilon^2\tau \qquad (3.8)$$

$$\frac{d\lambda}{du} = m + a^\circ\left[\frac{\varkappa^2}{\sigma^2}-(1+b^\circ)A^2\right] - \left(\frac{\varkappa^2}{\sigma^2}-A^2\right)\cos\lambda, \quad m=\frac{16(\beta-1)}{\varepsilon^2(1+2b\beta^2)}$$

$$a^\circ = \frac{[\beta+2a(2b\beta^2-1)]\sigma^2}{1+2b\beta^2}, \qquad b^\circ=\frac{\beta(4b\beta^2+\beta-2)}{[\beta+2a(2b\beta^2-1)]\sigma^2}$$

From (3.8) we find

$$\left\{m + a^\circ\left[\frac{\varkappa^2}{\sigma^2}-(1+b^\circ)A^2\right] - \left(\frac{\varkappa^2}{\sigma^2}-2A^2\right)\cos\lambda\right\}dA + \\ + {}^1/_2\,A\left(\frac{\varkappa^2}{\sigma^2}-A^2\right)\sin\lambda\,d\lambda = 0 \qquad (3.9)$$

This equation has the integral (3.10)

$${}^1/_2 mA^2 + {}^1/_2\,a^\circ\varkappa^2\sigma^{-2}A^2 - {}^1/_4 a^\circ(1+b^\circ)A^4 - {}^1/_2\,A^2(\varkappa^2\sigma^{-2}-A^2)\cos\lambda = -{}^1/_2\,c_0$$

Here c_0 is a constant of integration. The integral (3.10) can be written also as

$$(a^*-\cos\lambda)A^4-(b^*-e\cos\lambda)A^2=c_0 \qquad (3.11)$$

$$(a^* = {}^1/_2\,a^\circ(1+b^\circ),\ b^*=m+a^\circ e,\ e=\varkappa^2\sigma^{-2})$$

4. Let us investigate the phase trajectories for the autonomous system (3.8) in the XY-plane, for which $X=A\cos\lambda$, $Y=A\sin\lambda$, i.e. A and λ are natural polar coordinates. The phase trajectories are expressed by (3.10) or (3.11) and they are all symmetric relative to the X-axis. Because of (3.7) all the real trajectories lie on the boundary or within the circle $X^2+Y^2=e$. Let us first determine the singularities of the system (3.8). From the conditions

$$dA/du=0, \qquad d\lambda/du=0$$

we find the singular points

$$\text{a)}\quad \lambda_1=0, \qquad A_1=\left(\frac{m+e(a^\circ-1)}{a^\circ(1+b^\circ)-2}\right)^{1/2} \qquad (4.1)$$

$$\text{b)}\quad \lambda_2=\pi, \qquad A_2=\left(\frac{e(1+a^\circ)+m}{2+a^\circ(1+b^\circ)}\right)^{1/2} \qquad (4.2)$$

$$\text{c)}\quad A_3=\varkappa/\sigma, \quad \cos\lambda_3=a^\circ b^\circ-m/e \qquad (4.3)$$

The quantity m is proportional to resonance "detuning" $\beta - 1$. Let us show various cases of the phase trajectories as a function of m. They are associated with the singular points (a), (b) and (c) for which the value m is found in the respective intervals

a) $e(1 - a^\circ) < m < e(a^\circ b^\circ - 1)$

b) $e(1 + a^\circ) < m < e(a^\circ b^\circ - 1) \quad (e = \varkappa^2 / \sigma^2)$

c) $e(a^\circ b^\circ - 1) < m < e(a^\circ b^\circ + 1)$

Indeed, the point (c) is two points on the boundary circumference, which are symmetrically disposed relative to OX. The origin of reference is also singular since $dA/du \cdot = 0$ for $A = 0$.

Therefore, the following fundamental cases can be established.

1°. $m < -e(1 + a^\circ)$

In this case there exists just one singular point, the origin of reference, which is a center (Fig. 2), where the amplitude changes insignificantly

2°. $-e(1 + a^\circ) < m < e(1 - a^\circ)$

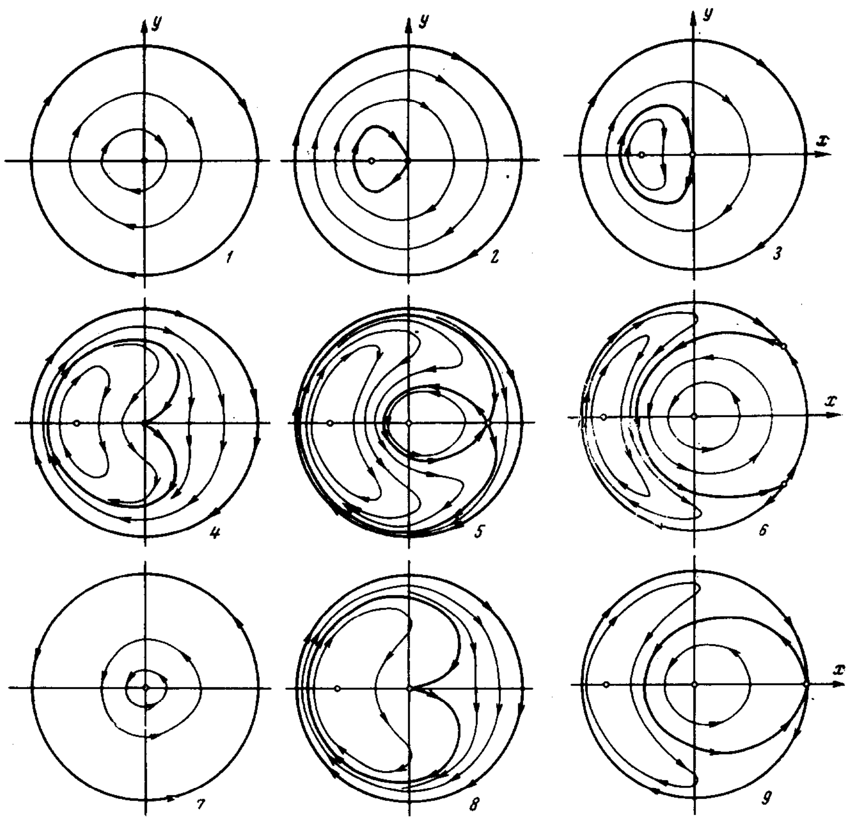

Fig. 2

Here two singular points exist, the origin and the point (b). In this case three different pictures of the phase trajectory behavior can be extracted:

$$2°.1. \qquad -e(1 + a°) < m < -ea°$$
$$2°.2. \qquad m = -ea°$$
$$2°.3. \qquad -ea° < m < e(1 - a°)$$

which are shown, respectively, in Figs. 2. 2–2. 4

$$3°. \qquad e(1 - a°) < m < e(a°b° - 1)$$

There are three singular points in this case, the point (b) and the reference origin, which are centers, and the point (a) which is a saddle point. The phase trajectories for this case are shown in Fig. 2. 5

$$4°. \qquad e(a°b° - 1) < m < e(a°b° + 1)$$

Four singular points exist, the point (b) and the origin which are centers, and the two points (c) on the boundary circumference, which are saddle points. The phase trajectories for this case are shown in Fig. 2. 6

$$5°. \qquad e(a°b° + 1) < m$$

Exactly as in the first case, we have just one singular point, the origin, which is a center. The phase trajectories for this case are shown in Fig. 2.7.

The transition from case 2° to case 3° and from case 3° to case 4° are manifested particularly clearly from the limiting values

$$m = e(1 - a°), \qquad m = e(a°b - 1)$$

which are shown in Figs. 2. 8 and 2. 9.

It results from Fig. 2 that for $m < e(1 + a°)$ the picture of the phase trajectories has no singularities, and for $m > -e(1 + a°)$ the point (b) is first isolated from the origin and moved to the left (Figs. 2. 2–2. 4). The point (b) will be a center, and the origin goes from a center to become a saddle point. After $m = e(1 - a°)$ the point (a), which moves to the right, is isolated from the origin. The origin again becomes a center, and the point (a) will be a saddle point (Fig. 2. 5). For $m = e(a°b° - 1)$ the point (a) reaches the boundary circumference and coincides with the points (c)(Fig. 2.9). For $m > e(a°b° + 1)$ the points (c) move from right to left along the boundary circumference and for $m = e(a°b° + 1)$ again coincide with the point (b) this time. Furthermore, for $m > e(a°b° + 1)$ the phase trajectory picture again has no singularities.

The phase trajectories yield a very clear picture of the system motion. It is seen that motions with constant amplitude are possible, points of center type correspond to them; motions with periodic oscillations in the amplitude are possible, closed phase trajectories correspond to them. Separatrices and singular saddle points correspond to transitional (nonperiodic) changes in the amplitude.

Energy transfer from one pendulum to the other can be observed in cases 2°–4°, the amplitude of oscillation of one pendulum diminishes considerably, while the amplitude of the oscillations of the other also increases significantly because of the dependence (3. 7).

5. Let us find the amplitude A as a function of the dimensionless time τ. We obtain from (3. 8) and (3. 11)

$$\frac{d\mu}{\pm\sqrt{(e-\mu)^2\mu^2 - (c_0 + b^*\mu - a^*\mu^2)^2}} = du \qquad (\mu = A^2) \tag{5.1}$$

Let us consider the polynomial

$$G(\mu) = (e - \mu)^2\mu^2 - (c_0 + b^*\mu - a^*\mu^2)^2 \tag{5.2}$$

The roots of the polynomial (5.2) coincide with the positive roots of (3.11) (for A^2) with $\cos\lambda = 1$ and $\cos\lambda = -1$. For different values of c_0 and m the polynomial (5.2) has four different roots or two real and two complex roots, i.e. it can be written

$$G(\mu) = (1 - a^*)\ (\mu - \mu_1)\ (\mu - \mu_2)\ (\mu - \mu_3)\ (\mu - \mu_4) \tag{5.3}$$

$$1 - a^* < 0,\ \ \mu_1 > \mu_2 > \mu_3 > \mu_4 > 0 \tag{5.4}$$

or as

$$G(\mu) = (1 - a^*)\ (\mu - \mu_1)\ (\mu - \mu_2)\ \ [(\mu - \nu)^2 + \omega^2]$$

$$\mu_1 > \mu_2 > 0, \qquad \nu = \frac{a^* b^* - e}{a^* - 1} - \frac{1}{2}(\mu_1 + \mu_2)$$

$$\omega^2 = \frac{c_0^2}{\mu_1\mu_2 (a^* - 1)} - \left[\frac{a^* b^* - e}{a^* - 1} - \frac{1}{2}(\mu_1 + \mu_2)\right] \quad (\omega > 0)$$

(ν and ω can still be obtained as complex roots of (3.11) for $\cos\lambda = \pm 1$).

The polynomial (5.2) will have the form (5.4) for some value of c_0, if for this value there exists just one phase trajectory which intersects the X-axis at points with the polar radii $A_1 = \sqrt{\mu_1}$ and $A_2 = \sqrt{\mu_2}$. The polynomial (5.2) will have the form (5.3) if for some value of c_0 there exist two phase trajectories which intersect the X-axis, the first at points with the polar radii $A_3 = \sqrt{\mu_3}$ and $A_4 = \sqrt{\mu_4}$, and the second at points with the polar radii $A_1 = \sqrt{\mu_1}$ and $A_2 = \sqrt{\mu_2}$.

The real roots are found directly while constructing the appropriate phase trajectories. In some cases, when the real roots are greater than e, real trajectories do not correspond to them.

It is easy to show that $G(\mu)$ has the form (5.4) in the cases 1° and 2° for all phase trajectories, and in case 4° for phase trajectories which close around the center (b), and also that the polynomial $G(\mu)$ has the form (5.3) in the case 5° for all phase trajectories and in case 4° for phase trajectories which close around the origin. Hence $\mu_1 > > \mu_2 > e$.

The situation is complicated somewhat for case 3°. Here the phase trajectory Φ_0, which is obtained for $c_0 = 0$ has an important part. For $m = m^\circ = {}^1/_2\, ea^\circ (b^\circ - 1)$ it coincides with the boundary circumference, for $m < m^\circ$ it is inside, and for $m > m^\circ$ outside the limits of this circumference.

The polynomial $G(\mu)$ has the form (5.4) for phase trajectories which close around the center (b) and for those which are located between the boundary circumference and Φ_0 (if the latter is within the boundary circumference).

The polynomial $G(\mu)$ has the form (5.3) for phase trajectories which close around the origin and for those which are located between the outer separatrix through the point (a) and Φ_0. If Φ_0 is outside the boundary circumference, it can happen that $\mu_1 > \mu_2 > e$.

Let us first examine the case when $G(\mu)$ has the form (5.3). We set

$$k^2 = \frac{(\mu_3 - \mu_4)(\mu_2 - \mu_1)}{(\mu_3 - \mu_1)(\mu_2 - \mu_4)}, \quad l^2 = \frac{4}{(\mu_1 - \mu_3)(\mu_2 - \mu_4)} \tag{5.5}$$

Then (utilizing [6], pp. 19–21), we obtain from (5.1) for μ in the range $\mu_4 \leqslant \mu \leqslant \mu_3$

$$\mu = A^2 = \frac{\mu_4(\mu_1 - \mu_3) + \mu_1(\mu_3 - \mu_4)\, sn^2 U}{\mu_1 - \mu_3 - (\mu_1 - \mu_4)\, sn^2 U} \qquad \left(U = \frac{\sqrt{a^* - 1}\,(u - u_0)}{l}\right) \tag{5.6}$$

where the modulus of the Jacobi elliptic function k is defined by (5.5), and u_0 is the value of the parameter u for $\mu = \mu_4$.

For μ in the interval $\mu_2 \leqslant \mu \leqslant \mu_1$ we have

$$\mu = \frac{\mu_2(\mu_1-\mu_3)-\mu_3(\mu_1-\mu_2)\,sn^2U}{\mu_1-\mu_3+(\mu_3-\mu_4)\,sn^2\,U} \tag{5.7}$$

where the modulus k has the same value as in (5.6), and u_0 is the value of the parameter u for $\mu = \mu_2$.

The period of long-period oscillations in the amplitude A with respect to the time τ is defined by the formula

$$\varepsilon^2 T = \frac{16\beta l}{(1+2b\beta^2)\sqrt{a^*-1}}\,K(k) \tag{5.8}$$

Here $K(k)$ is the complete elliptic integral of the first kind in Legendre form of modulus k. The expression (5.8) explicitly confirms the slow change in the amplitude A (the amplitudes B, and phases φ and ψ, respectively). It is seen that the period of variation of A for the two cases (5.6) and (5.7) is identical although the motions themselves are completely distinct.

Let us now examine the case when $G(\mu)$ has the form (5.4). Here we use the notation ([6], pp. 19–21)

$$\mathrm{tg}\,p = \frac{\mu_1-\nu}{\omega}, \quad \mathrm{tg}\,q = \frac{\mu_2-\nu}{\omega} \tag{5.9}$$

For μ in the range $\mu_2 \leqslant \mu \leqslant \mu_1$ we obtain

$$\mu = \frac{\mu_1\cos p+\mu_2\cos q+(\mu_1\cos p-\mu_2\cos q)\,cnU}{\cos p+\cos q+(\cos p-\cos q)\,cnU} \tag{5.10}$$

where the modulus k of the Jacobi elliptic function and the quantity l are defined by the expressions

$$k^2 = \sin^2\frac{p-q}{2}, \quad l = -\frac{(\cos p\cos q)^{1/2}}{\omega} \tag{5.11}$$

and u_0 is the value of the parameter u for $\mu = \mu_1$.

In this case the period of oscillations of the amplitude A is again defined by (5.8), with the sole difference that k and l have the values (5.11).

After having determined A as a function of τ by utilizing (3.7) we can also determine B as a function of τ.

It must be noted that the obtained resonance solution of the system (1.7) is also valid for the nonresonance case $\beta \neq 1$.

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